

# ON THE TOPOLOGY OF A RESOLUTION OF ISOLATED SINGULARITIES

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**ABSTRACT.** Let  $Y$  be a complex projective variety of dimension  $n$  with isolated singularities,  $\pi : X \rightarrow Y$  a resolution of singularities,  $G := \pi^{-1}\text{Sing}(Y)$  the exceptional locus. From Decomposition Theorem one knows that the map  $H^{k-1}(G) \rightarrow H^k(Y, Y \setminus \text{Sing}(Y))$  vanishes for  $k > n$ . Assuming this vanishing, we give a short proof of Decomposition Theorem for  $\pi$ . A consequence is a short proof of the Decomposition Theorem for  $\pi$  in all cases where one can prove the vanishing directly. This happens when either  $Y$  is a normal surface, or when  $\pi$  is the blowing-up of  $Y$  along  $\text{Sing}(Y)$  with smooth and connected fibres, or when  $\pi$  admits a natural Gysin morphism. We prove that this last condition is equivalent to say that the map  $H^{k-1}(G) \rightarrow H^k(Y, Y \setminus \text{Sing}(Y))$  vanishes for any  $k$ , and that the pull-back  $\pi_k^* : H^k(Y) \rightarrow H^k(X)$  is injective. This provides a relationship between Decomposition Theorem and Bivariant Theory.

*Keywords:* Projective variety, Isolated singularities, Resolution of singularities, Derived category, Intersection cohomology, Decomposition Theorem, Bivariant Theory, Gysin morphism, Cohomology manifold.

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## 1. INTRODUCTION

Consider a  $n$ -dimensional complex projective variety  $Y$  with *isolated singularities*. Fix a *desingularization*  $\pi : X \rightarrow Y$  of  $Y$ . This paper is addressed at the study of some topological properties of the map  $\pi$ . In a previous paper [14] we already observed that, even though  $\pi$  is never a *local complete intersection* map, in some very special case it may anyway admit a *natural Gysin morphism*. By natural Gysin morphism we mean a *topological bivariant class* [20, §7], [7]

$$\theta \in T^0(X \xrightarrow{\pi} Y) := \text{Hom}_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y),$$

commuting with restrictions to the smooth locus of  $Y$  (here and in the following  $D^b(Y)$  denotes the *bounded derived category* of sheaves of  $\mathbb{Q}$ -vector spaces on  $Y$ ).

In this paper we give a complete characterization of morphisms like  $\pi$  admitting a natural Gysin morphism by means of the *Decomposition Theorem* [2], [6], [8], [9]. In some sense, what we are going to prove is that  $\pi$  admits a natural Gysin morphism if and only if  $Y$  is a  *$\mathbb{Q}$ -intersection cohomology manifold*, i.e.  $IC_Y^\bullet \simeq \mathbb{Q}_Y[n]$  in  $D^b(Y)$  ( $IC_Y^\bullet$  denotes *intersection cohomology complex* of  $Y$  [17, p. 156], [27]).

Furthermore, in this case, there is a unique natural Gysin morphism  $\theta$ , and it arises from the Decomposition Theorem (compare with Theorem 1.2 below).

The Decomposition Theorem is a beautiful and very deep result about algebraic maps. In the words of MacPherson “it contains as special cases the deepest homological properties of algebraic maps that we know” [26], [34]. As observed in [34, Remark 2.14], since the proof of the Decomposition Theorem proceeds by induction on the dimension of the strata of the singular locus, a key point of such a Theorem is the case of varieties with isolated singularities:

**Theorem 1.1** (Decomposition Theorem for varieties with isolated singularities). *In  $D^b(Y)$  we have a decomposition*

$$R\pi_*\mathbb{Q}_X \cong IC_Y^\bullet[-n] \oplus \mathcal{H}^\bullet$$

where  $\mathcal{H}^\bullet$  is quasi isomorphic to a skyscraper complex on  $\text{Sing}(Y)$  and

- (1)  $\mathcal{H}^k(\mathcal{H}^\bullet) \cong H^k(G)$ , for any  $k \geq n$ ,
- (2)  $\mathcal{H}^k(\mathcal{H}^\bullet) \cong H_{2n-k}(G)$ , for any  $k < n$ ,

where we set  $G := \pi^{-1}(\text{Sing}(Y))$ .

The relationship between Gysin morphism and Decomposition Theorem is mostly related to an important topological property of the morphism  $\pi$ . Specifically, in [22] and [32] it is showed that Theorem 1.1 implies the following vanishing

$$(1) \quad H^{k-1}(G) \rightarrow H^k(Y, U) \text{ vanishes for } k > n.$$

One of the main points we would like to stress in this paper (compare with Theorem 3.1) is that

*the vanishing (1) is equivalent to the Decomposition Theorem.*

More precisely, what we are going to do in this paper is to prove that assuming (1), one can prove Theorem 1.1 in few pages. Actually this equivalence is already implicit in the argument developed by Navarro Aznar in order to prove [30, (6.3) Corollaire, p. 293]. In fact, after proving (1) using Hodge Theory, in [30] one proves relative Hard Lefschetz Theorem and concludes thanks to Deligne’s Theorems on degeneration of spectral sequences. Instead, here we give a more simple and direct proof, without using Hard Lefschetz Theorem. In fact we deduce the splitting in derived category by a simple result about short exact sequences of complexes (compare with Lemma 4.7).

A byproduct of our result is a short proof of the Decomposition Theorem in all cases where one can prove property (1) directly. This happens when either  $2 \dim G < n$  (for trivial reasons), or when  $Y$  is a *normal surface* in view of *Mumford’s Theorem* [23], [29], or when  $\pi : X \rightarrow Y$  is the *blowing-up* of  $Y$  along  $\text{Sing}(Y)$  with smooth and connected fibres (see Remark 5.1). It is worth remarking that if  $Y$  is locally complete intersection then *Milnor’s Theorem* on the connectivity of the *link* [16] implies (via Lemma 4.1 below) that the map  $H^{k-1}(G) \rightarrow H^k(Y, U)$  vanishes for any  $k \geq n+2$ . Therefore in this case the question reduces to only check that

the map  $H^n(G) \rightarrow H^{n+1}(Y, U)$  vanishes. This in turn is equivalent to require that  $H_n(G)$ , which is contained in  $H_n(X)$  via push-forward, is a non degenerate subspace of  $H_n(X)$  with respect to the natural intersection form  $H_n(X) \times H_n(X) \rightarrow H_0(X)$  (see Remark 5.1, (i)). Another case is when  $\pi$  admits a Gysin morphism. Indeed, in this case it is very easy to prove the stronger property

$$H^{k-1}(G) \rightarrow H^k(Y, U) \text{ vanishes for } k > 0.$$

This is the real reason why in our approach the same line of arguments leads to both Theorem 1.1 and the following:

**Theorem 1.2.** *There exists a natural Gysin morphism for  $\pi$  if and only if  $Y$  is a  $\mathbb{Q}$ -intersection cohomology manifold. In this case, in  $D^b(Y)$  we have a decomposition*

$$R\pi_*\mathbb{Q}_X \cong IC_Y^\bullet[-n] \oplus \mathcal{H}^\bullet \cong \mathbb{Q}_Y \oplus \bigoplus_{k \geq 1} R^k\pi_*\mathbb{Q}_X[-k].$$

Moreover a natural Gysin morphism is unique, and, up to multiplication by a nonzero rational number, it comes from the decomposition above via projection onto  $\mathbb{Q}_Y$ .

For a more precise and complete statement see Theorem 3.2 and Remark 3.3 below. For instance, from Theorem 3.2, (ix), we see that natural Gysin morphisms occur when  $Y$  is nodal of even dimension  $n$ , or when  $Y$  is a cone over a smooth basis  $M$  with  $H^\bullet(M) \cong H^\bullet(\mathbb{P}^{n-1})$ . We stress that the existence of a natural Gysin morphism forces the exceptional locus  $G$  to have dimension 0 or  $n - 1$  (see Remark 6.1).

Last but not least, we have been led to consider the issues addressed in this paper by our previous work on Noether-Lefschetz Theory. We refer to the papers [10], [11], [12], [13] anyone interested in the overlaps between the topological properties investigated here and Noether-Lefschetz Theorem (specifically, we made an heavy use of Decomposition Theorem in [12, Remark 3 and Theorem 6, (6.3), p. 169], and in [13, Theorem 2.1, proof of (a), p. 262]).

## 2. NOTATIONS

(i) Let  $Y$  be a complex irreducible projective variety of dimension  $n \geq 1$ , with isolated singularities. Let  $\pi : X \rightarrow Y$  be a resolution of the singularities of  $Y$ . For any  $y \in \text{Sing}(Y)$  set  $G_y := \pi^{-1}(y)$ . Set  $G := \bigcup_{y \in \text{Sing}(Y)} G_y = \pi^{-1}(\text{Sing}(Y))$ . Let  $i : G \hookrightarrow X$  be the inclusion.

(ii) All cohomology and homology groups are with  $\mathbb{Q}$ -coefficients.

(iii) Set  $U := Y \setminus \text{Sing}(Y) \cong X \setminus G$ . Denote by  $\alpha : U \hookrightarrow Y$  and  $\beta : U \hookrightarrow X$  the inclusions. For any integer  $k$  we have the following natural commutative diagram:

$$(2) \quad \begin{array}{ccc} H^k(Y) & \xrightarrow{\pi_k^*} & H^k(X) \\ \alpha_k^* \searrow & & \swarrow \beta_k^* \\ & H^k(U) & \end{array}$$

where all the maps denote pull-back.

*Remark 2.1.* From commutativity of diagram (2) we get  $\mathfrak{S}(\alpha_k^*) \subseteq \mathfrak{S}(\beta_k^*)$ . Since  $H^k(Y) \cong H^k(X)$  for  $k \leq 0$  or  $k \geq 2n$ , we have  $\mathfrak{S}(\alpha_k^*) = \mathfrak{S}(\beta_k^*)$  for  $k \leq 0$  or  $k \geq 2n$ . It may happen that  $\mathfrak{S}(\alpha_k^*) \neq \mathfrak{S}(\beta_k^*)$ . We may interpret the condition  $\mathfrak{S}(\alpha_k^*) = \mathfrak{S}(\beta_k^*)$  as follows. From Universal Coefficient Theorem and Lefschetz Duality Theorem [31, p. 248 and p. 297] we have  $H^k(U) \cong H_{2n-k}(Y, \text{Sing}(Y))$  for any  $k$ . Since  $\text{Sing}(Y)$  is finite we also have  $H_{2n-k}(Y) \cong H_{2n-k}(Y, \text{Sing}(Y))$  for  $k \leq 2n - 2$ , and  $H_1(Y) \subseteq H_1(Y, \text{Sing}(Y))$ . Therefore, for  $k \leq 2n - 2$ , diagram (2) identifies with the diagram:

$$\begin{array}{ccc} H^k(Y) & \longrightarrow & H_{2n-k}(X) \\ & \searrow & \swarrow \\ & H_{2n-k}(Y) & \end{array}$$

where the map  $H^k(Y) \rightarrow H_{2n-k}(X)$  is the composite of Poincaré Duality  $H^k(X) \cong H_{2n-k}(X)$  with the pull-back  $\pi_k^*$ , the map  $H_{2n-k}(X) \rightarrow H_{2n-k}(Y)$  is the push-forward, and the map  $H^k(Y) \xrightarrow{\cdot \cap [Y]} H_{2n-k}(Y)$  is the *duality morphism*, i.e. the cap-product with the fundamental class  $[Y] \in H_{2n}(Y)$  [28]. It follows that  $\mathfrak{S}(\alpha_k^*) = \mathfrak{S}(\beta_k^*)$  if and only if any cycle in  $H_{2n-k}(Y)$  coming from  $H_{2n-k}(X)$  via push-forward is the cap-product of a cocycle in  $H^k(Y)$  with the fundamental class  $[Y]$ . This holds true also for  $k = 2n - 1$  because  $H_1(Y) \subseteq H_1(Y, \text{Sing}(Y)) \cong H^{2n-1}(U)$ .

(iv) Embed  $Y$  in some projective space  $\mathbb{P}^N$ . For any  $y \in \text{Sing}(Y)$  choose a small closed ball  $S_y \subset \mathbb{P}^N$  around  $y$ , and set  $B_y := S_y \cap Y$ ,  $D_y := \pi^{-1}(B_y)$ ,  $B := \bigcup_{y \in \text{Sing}(Y)} B_y$ , and  $D := \pi^{-1}(B)$ .  $B_y$  is homeomorphic to the cone over the link  $\partial B_y$  of the singularity  $y \in Y$ , with vertex at  $y$  [16, p. 23].  $B_y$  is contractible, by excision we have  $H^k(Y, U) \cong H^k(B, B \setminus \text{Sing}(Y)) \cong H^k(B, \partial B)$  for any  $k$ , and from the cohomology long exact sequence of the pair  $(B, \partial B)$  we get  $H^k(Y, U) \cong H^{k-1}(\partial B)$  for any  $k \geq 2$ . We have  $\partial D \cong \partial B$  via  $\pi$ , and by excision we have  $H^k(X, U) \cong H^k(D, D \setminus G) \cong H^k(D, \partial D)$  for any  $k$  [17, p. 38]. Since  $G$  is homotopy equivalent to  $D$ , we have  $H^k(G) \cong H^k(D)$ . Putting all together, from the cohomology long exact sequence of the pair  $(D, \partial D)$  we get the following exact sequence

$$(3) \quad H^k(X, U) \rightarrow H^k(G) \rightarrow H^{k+1}(Y, U) \xrightarrow{\gamma_{k+1}^*} H^{k+1}(X, U)$$

for any  $k \geq 1$ , where  $\gamma_{k+1}^*$  denotes the pull-back. Observe that since  $\text{Sing}(Y)$  is finite we have  $H^k(G) = \bigoplus_{y \in \text{Sing}(Y)} H^k(G_y)$ ,  $H^k(B) = \bigoplus_{y \in \text{Sing}(Y)} H^k(B_y)$ ,  $H^k(\partial B) = \bigoplus_{y \in \text{Sing}(Y)} H^k(\partial B_y)$ .

*Remark 2.2.* Assume  $Y$  is locally complete intersection. In this case, from the connectivity of the link [16, Milnor's Theorem p. 76, and Hamm's Theorem p. 80], it follows that the duality morphism  $H^k(Y) \rightarrow H_{2n-k}(Y)$  is an isomorphism for any  $k \notin \{n-1, n, n+1\}$ , is injective for  $k = n-1$ , and is surjective for  $k = n+1$ . In particular  $\mathfrak{S}(\alpha_k^*) = \mathfrak{S}(\beta_k^*)$  for any  $k \notin \{n-1, n\}$ . In order to prove this property, we argue as follows. We may assume  $0 < k < 2n$  and  $n \geq 2$ . From the cohomology long exact sequence of the pair  $(Y, U)$  we have:

$$(4) \quad \dots \rightarrow H^k(Y, U) \rightarrow H^k(Y) \rightarrow H^k(U) \rightarrow H^{k+1}(Y, U) \rightarrow \dots,$$

and by excision  $H^k(Y, U) \cong H^k(B, \partial B)$ . Taking into account that each  $B_y$  is contractible and that  $\partial B_y$  is path connected [16, loc. cit.], from the cohomology long exact sequence of the pair  $(B, \partial B)$  we get  $H^1(B, \partial B) = 0$  and  $H^k(B, \partial B) \cong H^{k-1}(\partial B)$  for  $k \geq 2$ . Since  $H^k(U) \cong H_{2n-k}(Y, \text{Sing}(Y))$ , and  $H_{2n-k}(Y) \cong H_{2n-k}(Y, \text{Sing}(Y))$  for  $k \leq 2n-2$ , from (4) we get the exact sequence for  $k \notin \{1, 2n-1\}$  (compare with [15, p. 5]):

$$H^{k-1}(\partial B) \rightarrow H^k(Y) \rightarrow H_{2n-k}(Y) \rightarrow H^k(\partial B).$$

Each  $\partial B_y$  is  $(n-2)$ -connected by Milnor's Theorem [16, loc. cit.], and it is a compact oriented real manifold of dimension  $2n-1$ , in particular  $h^k(\partial B_y) = h^{2n-1-k}(\partial B_y)$  by Poincaré Duality [16, p. 91]. It follows that the map  $H^k(Y) \rightarrow H_{2n-k}(Y)$  is an isomorphism for  $k \notin \{1, n-1, n, n+1, 2n-1\}$ . As for the case  $k = 1 \neq n-1$ , this follows from (4) because  $H^1(Y, U) \cong H^1(B, \partial B) = 0$ ,  $H^1(U) \cong H_{2n-1}(Y, \text{Sing}(Y)) \cong H_{2n-1}(Y)$ , and  $H^2(Y, U) \cong H^2(B, \partial B) \cong H^1(\partial B) = 0$  by connectivity of the link. When  $k = 2n-1 \neq n+1$  we have  $H^{2n-1}(Y, U) \cong H^{2n-1}(B, \partial B) = H^{2n-2}(\partial B) = 0$ , therefore  $H^{2n-1}(Y) \hookrightarrow H^{2n-1}(U)$ . On the other hand  $H_1(Y) \hookrightarrow H_1(Y, \text{Sing}(Y)) \cong H^{2n-1}(U)$ . It follows that the duality morphism  $H^{2n-1}(Y) \rightarrow H_1(Y)$  is injective. Then it is an isomorphism because we have just seen, in the case  $k = 1$ , that  $h^1(Y) = h_{2n-1}(Y)$ . Finally notice that, when  $n \geq 3$ , from previous analysis and (4) we get the exact sequence:

$$\begin{aligned} 0 \rightarrow H^{n-1}(Y) \rightarrow H_{n+1}(Y) \rightarrow H^{n-1}(\partial B) \rightarrow H^n(Y) \rightarrow H_n(Y) \\ \rightarrow H^n(\partial B) \rightarrow H^{n+1}(Y) \rightarrow H_{n-1}(Y) \rightarrow 0. \end{aligned}$$

Therefore the duality morphism  $H^{n-1}(Y) \rightarrow H_{n+1}(Y)$  is injective, and  $H^{n+1}(Y) \rightarrow H_{n-1}(Y)$  is onto. This holds true also when  $n = 2$ . In fact also in this case we have  $H^1(B, \partial B) = 0$ , which implies that the duality morphism  $H^1(Y) \rightarrow H_3(Y)$  is injective. Moreover a similar analysis as before shows that the image of  $H^3(Y)$  and  $H_1(Y)$  have the same codimension in  $H^3(U)$ , and therefore they are equal. This concludes the proof of the claim.

(v) By [31, Lemma 14, p. 351] we have  $H^k(X, U) \cong H_{2n-k}(G)$ . Therefore from the cohomology long exact sequence of the pair  $(X, U)$  we get a long exact sequence:

$$(5) \quad \dots \rightarrow H^{k-1}(U) \rightarrow H_{2n-k}(G) \rightarrow H^k(X) \xrightarrow{\beta_k^*} H^k(U) \rightarrow \dots$$

(vi) For any  $y \in \text{Sing}(Y)$  set:

$$H_y^k := \begin{cases} H^k(G_y) & \text{if } k \geq n \\ H_{2n-k}(G_y) & \text{if } k < n. \end{cases}$$

Let  $\mathcal{H}_y^k$  be the skyscraper sheaf on  $Y$  with stalk at  $y$  given by  $H_y^k$ . Set  $H^k := \bigoplus_{y \in \text{Sing}(Y)} \mathcal{H}_y^k$  and  $\mathcal{H}^k := \bigoplus_{y \in \text{Sing}(Y)} \mathcal{H}_y^k$ . We consider  $\mathcal{H}^\bullet$  as a complex of sheaves on  $Y$  with vanishing differentials  $d_{\mathcal{H}^\bullet}^k = 0$ .

*Remark 2.3.* By Universal Coefficient Theorem [31, p. 248] it follows that the  $\mathbb{Q}$ -vector spaces  $H^{n-k}$  and  $H^{n+k}$  are isomorphic for any  $k$ . This implies that  $\mathcal{H}^\bullet[n]$  is self-dual, i.e. in the bounded derived category  $D^b(Y)$  of  $Y$  we have  $\mathcal{H}^\bullet[n] \cong D(\mathcal{H}^\bullet[n])$ . Taking into account that in  $\mathcal{H}^\bullet[n]$  all the differentials vanish, to prove that  $\mathcal{H}^\bullet[n]$  is self-dual it suffices to prove that the complexes  $\mathcal{H}^\bullet[n]$  and  $D(\mathcal{H}^\bullet[n])$  have isomorphic sheaf cohomology. Since  $\mathcal{H}^\bullet[n]$  is supported on a finite set, this amounts to prove that  $\mathcal{H}^\bullet[n]$  and  $D(\mathcal{H}^\bullet[n])$  have isomorphic hypercohomology, i.e. that

$$\mathbb{H}^k(\mathcal{H}^\bullet[n]) \cong \mathbb{H}^k(D(\mathcal{H}^\bullet[n]))$$

for any  $k$ . But by Poincaré-Verdier Duality [17, p. 69, Theorem 3.3.10] we have:

$$\mathbb{H}^k(D(\mathcal{H}^\bullet[n])) \cong \mathbb{H}^{-k}(\mathcal{H}^\bullet[n])^\vee \cong \mathbb{H}^{n-k}(\mathcal{H}^\bullet)^\vee \cong (H^{n-k})^\vee \cong H^{n+k} \cong \mathbb{H}^k(\mathcal{H}^\bullet[n]).$$

(vii) We say that a graded morphism  $\theta_\bullet : H^\bullet(X) \rightarrow H^\bullet(Y)$  is *natural* if for any  $k$  one has  $\theta_k \circ \pi_k^* = \text{id}_{H^k(Y)}$ , and the following diagram commutes [14]:

$$\begin{array}{ccc} H^k(Y) & \xleftarrow{\theta_k} & H^k(X) \\ \alpha_k^* \searrow & & \swarrow \beta_k^* \\ & H^k(U), & \end{array}$$

i.e.  $\alpha_k^* \circ \theta_k = \beta_k^*$ .

*Remark 2.4.* The existence of a natural graded morphism  $\theta_\bullet : H^\bullet(X) \rightarrow H^\bullet(Y)$  is equivalent to say that, for any  $k$ , the pull-back  $\pi_k^* : H^k(Y) \rightarrow H^k(X)$  is injective and  $\Im(\alpha_k^*) = \Im(\beta_k^*)$  (compare with the proof of (i)  $\implies$  (ii) in Theorem 3.2 below).

(viii) We say that a (topological) bivariant class  $\theta \in \text{Hom}_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y)$  is *natural* if the induced graded morphism  $\theta_\bullet : H^\bullet(X) \rightarrow H^\bullet(Y)$  is natural [14], [20].

*Remark 2.5.* Fix any bivariant class  $\theta \in H^0(X \xrightarrow{\pi} Y) \cong \text{Hom}_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y)$ . Let  $\theta_0 : H^0(X) \rightarrow H^0(Y)$  be the induced map. Let  $q \in \mathbb{Q}$  be such that  $\theta_0(1_X) = q \cdot 1_Y \in H^0(Y) \cong \mathbb{Q}$  [31, p. 238]. Put

$$\deg \theta := q.$$

For any  $k$  and any  $c \in H^k(Y)$ , by the projection formula [20, (G<sub>4</sub>), (i), p. 26], and [31, 9, p. 251], we have :

$$(6) \quad \theta_k(\pi_k^*(c)) = \theta_k(1_X \cup \pi_k^*(c)) = \theta_0(1_X) \cup c = \deg \theta \cdot (1_Y \cup c) = \deg \theta \cdot c.$$

It follows that for any  $k$  one has:

$$(7) \quad \theta_k \circ \pi_k^* = \deg \theta \cdot \text{id}_{H^k(Y)}.$$

Next consider the independent square:

$$\begin{array}{ccc} U & \xrightarrow{\beta} & X \\ \parallel & & \pi \downarrow \\ U & \xrightarrow{\alpha} & Y \end{array}$$

and set  $\theta' := \alpha^*(\theta) \in \text{Hom}_{D^b(U)}(\mathbb{Q}_U, \mathbb{Q}_U)$  [20, (G<sub>2</sub>), p. 26]. Applying [20, (G<sub>2</sub>), (ii), p. 26] to the square:

$$\begin{array}{ccc} H^0(U) & \xleftarrow{\beta_0^*} & H^0(X) \\ \theta'_0 \downarrow & & \theta_0 \downarrow \\ H^0(U) & \xleftarrow{\alpha_0^*} & H^0(Y) \end{array}$$

we get

$$\theta'_0(1_U) = \theta'_0(\beta_0^*(1_X)) = \beta_0^*(\theta_0(1_X)) = \beta_0^*(\deg \theta \cdot 1_Y) = \deg \theta \cdot \beta_0^*(1_Y) = \deg \theta \cdot 1_U.$$

Since  $\pi|_U = \text{id}_U$ , as in (6) we deduce for any  $k$  and any  $c \in H^k(U)$ :

$$\theta'_k(c) = \theta'_k((\pi|_U)^*(c)) = \theta'_k(1_U \cup c) = \theta'_0(1_U) \cup c = \deg \theta \cdot (1_U \cup c) = \deg \theta \cdot c,$$

i.e.

$$(8) \quad \theta'_k = \deg \theta \cdot \text{id}_{H^k(U)}.$$

From [20, (G<sub>2</sub>), (ii), p. 26] it follows that

$$(9) \quad \deg \theta \cdot \beta_k^* = \theta'_k \circ \beta_k^* = \alpha_k^* \circ \theta_k$$

for any  $k$ . By (7) and (9) we see that *a bivariant class  $\theta$  is natural if and only if  $\deg \theta = 1$ , and this is equivalent to say that  $\beta_k^* = \alpha_k^* \circ \theta_k$  for any  $k$* . Observe that if  $\theta$  is any bivariant class with  $\deg \theta \neq 0$ , then  $\frac{1}{\deg \theta} \theta$  is natural.

(ix) We say that  $Y$  is a  $\mathbb{Q}$ -cohomology (or homology) manifold if for any  $y \in Y$  and any  $k \neq 2n$  one has  $H^k(Y, Y \setminus \{y\}) = 0$ , and  $H^{2n}(Y, Y \setminus \{y\}) \cong \mathbb{Q}$  [27], [28]. Recall that  $Y$  is a  $\mathbb{Q}$ -intersection cohomology manifold if  $IC_Y^\bullet \cong \mathbb{Q}_Y[n]$  in  $D^b(Y)$ , where  $IC_Y^\bullet$  denotes the intersection cohomology complex of  $Y$  [17, p. 156], [27].

*Remark 2.6.* By [20, 3.1.4, p. 34] we know that there is a mapping  $\phi : X \rightarrow \mathbb{R}^m$  such that  $(\pi, \phi) : X \rightarrow Y \times \mathbb{R}^m$  is a closed imbedding. In this case one has

$$H^0(X \xrightarrow{\pi} Y) \cong H^m(Y \times \mathbb{R}^m, Y \times \mathbb{R}^m \setminus X_\phi),$$

where  $X_\phi$  is the image of  $X$  in  $Y \times \mathbb{R}^m$ . If  $Y$  is a  $\mathbb{Q}$ -cohomology manifold, then by Poincaré-Alexander-Lefschetz Duality [1, Theorem 1.1] we have:

$$H^m(Y \times \mathbb{R}^m, Y \times \mathbb{R}^m \setminus X_\phi) \cong H_{2n}(X).$$

It follows that

$$(10) \quad \dim_{\mathbb{Q}} H^0(X \xrightarrow{\pi} Y) = 1.$$

On the other hand, since  $U$  is smooth, we also have [19, Lemma 2 and (26), p. 217]:

$$H^0(U \xrightarrow{\text{id}_U} U) \cong H^m(U \times \mathbb{R}^m, U \times \mathbb{R}^m \setminus U_\phi) \cong H_{2n}^{BM}(U) \cong H^0(U) \cong \mathbb{Q},$$

where  $H_{2n}^{BM}(U)$  denotes Borel-Moore homology. Therefore the pull-back

$$\alpha^* : H^0(X \xrightarrow{\pi} Y) \rightarrow H^0(U \xrightarrow{\text{id}_U} U)$$

for bivariant classes identifies with the restriction in Borel-Moore homology:

$$H_{2n}(X) \cong H_{2n}^{BM}(U).$$

Comparing with (8) and (10), this proves that *if  $Y$  is a  $\mathbb{Q}$ -cohomology manifold then there is a unique natural bivariant class.*

(x) Let  $\mathcal{I}^\bullet$  be an injective resolution of  $\mathbb{Q}_X$ . Let  $\mathcal{J}^\bullet := \pi_*(\mathcal{I}^\bullet)$  be the derived direct image  $R\pi_*\mathbb{Q}_X$  of  $\mathbb{Q}_X$  in  $D^b(Y)$ . When  $k \geq 1$  the cohomology sheaves  $R^k\pi_*\mathbb{Q}_X = H^k(\mathcal{J}^\bullet)$  are supported on  $\text{Sing}(Y)$ , and for any  $y \in \text{Sing}(Y)$  we have  $H^k(\mathcal{J}^\bullet)_y = H^k(G_y)$ .

*Remark 2.7.* The complex  $\mathcal{J}^\bullet[n]$  is self-dual. In fact by [17, p. 69, Proposition 3.3.7, (ii)] we have:

$$D(\mathcal{J}^\bullet[n]) = D(R\pi_*\mathbb{Q}_X[n]) = R\pi_*(D(\mathbb{Q}_X[n])) = R\pi_*(\mathbb{Q}_X[n]) = \mathcal{J}^\bullet[n].$$

(xi) Since  $Y$  has only isolated singularities, we have [17, Proposition 5.4.4, p. 157]:

$$(11) \quad IH^k(Y) \cong \begin{cases} H^k(Y) & \text{if } k > n \\ \mathfrak{S}(\alpha_n^*) & \text{if } k = n \\ H^k(U) & \text{if } k < n. \end{cases}$$

### 3. THE MAIN RESULTS

Theorem 3.1 below is essentially already known. Property (i) implies (ii) by [32, Theorem 1.11, p. 518]. That property (ii) implies (i) is implicit in the argument developed by Navarro in order to prove [30, (6.3) Corollaire, p. 293] using a relative version of Hard Lefschetz Theorem. Here we give a more simple and direct proof that (ii) implies (i), without using Hard Lefschetz Theorem.

**Theorem 3.1.** *The following properties are equivalent.*

- (i) *In the derived category of  $Y$  there is an isomorphism  $R\pi_*\mathbb{Q}_X \cong IC_Y[n] \oplus \mathcal{H}^\bullet$ .*
- (ii) *The map  $H^{k-1}(G) \rightarrow H^k(Y, U)$  vanishes for any  $k > n$ .*

The equivalence of properties (v), (vi) and (vii) in next Theorem 3.2 are already known [4], [28], [27]. We insert them in the claim for Reader's convenience. We refer to [27] for other equivalence concerning a  $\mathbb{Q}$ -cohomology manifold.

**Theorem 3.2.** *The following properties are equivalent.*

- (i) *The map  $H^{k-1}(G) \rightarrow H^k(Y, U)$  vanishes for any  $k > 0$  and the pull-back  $\pi_k^*$  is injective.*
- (ii) *There exists a natural graded morphism  $\theta_\bullet : H^\bullet(X) \rightarrow H^\bullet(Y)$ .*
- (iii) *There exists a natural bivariant class  $\theta \in \text{Hom}_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y)$ .*
- (iv) *The natural map  $H^\bullet(Y) \rightarrow IH^\bullet(Y)$  is an isomorphism;*
- (v)  *$Y$  is a  $\mathbb{Q}$ -intersection cohomology manifold.*
- (vi)  *$Y$  is a  $\mathbb{Q}$ -cohomology manifold.*
- (vii) *The duality morphism  $H^\bullet(Y) \xrightarrow{\cdot \cap [Y]} H_{2n-\bullet}(Y)$  is an isomorphism (i.e.  $Y$  satisfies Poincaré Duality).*



(viii) In  $D^b(Y)$  there exists a decomposition

$$(12) \quad R\pi_*\mathbb{Q}_X \cong \mathbb{Q}_Y \oplus \bigoplus_{k \geq 1} R^k\pi_*\mathbb{Q}_X[-k].$$

Moreover, if  $\pi : X \rightarrow Y$  is the blowing-up of  $Y$  along  $\text{Sing}(Y)$  with smooth and connected fibres, then previous properties are equivalent to the following property:

(ix) For any  $y \in \text{Sing}(Y)$  one has  $H^\bullet(G_y) \cong H^\bullet(\mathbb{P}^{n-1})$ .

*Remark 3.3.* (i) Projecting onto  $\mathbb{Q}_Y$ , from the decomposition (12) we obtain a bivariant class  $\eta \in \text{Hom}_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y)$ , whose induced Gysin morphisms  $\eta_k : H^k(X) \rightarrow H^k(Y)$  are surjective. In particular  $\deg \eta \neq 0$ . By Remark 2.6 it follows that  $\frac{1}{\deg \eta} \eta$  is the unique natural bivariant class.

(ii) The natural morphism  $\theta_\bullet : H^\bullet(X) \rightarrow H^\bullet(Y)$  is unique and identifies with the push-forward via Poincaré Duality:

$$H^\bullet(X) \cong H_{2n-\bullet}(X) \rightarrow H_{2n-\bullet}(Y) \cong H^\bullet(Y).$$

In fact by Remark 2.1 we know that, for  $k < 2n - 1$ , the restriction map  $\alpha_k^* : H^k(Y) \rightarrow H^k(U)$  is nothing but the duality (iso)morphism because  $H^k(U) \cong H_{2n-k}(Y)$ . Therefore  $\theta_k = (\alpha_k^*)^{-1} \circ \beta_k^*$ . The case  $k = 2n - 1$  is similar because  $H_1(Y) \subseteq H^{2n-1}(U)$  (again compare with Remark 2.1).

#### 4. PRELIMINARIES

**Lemma 4.1.** *The following sequences are exact:*

$$\begin{aligned} 0 \rightarrow H^k(Y) &\xrightarrow{\pi_k^*} H^k(X) \xrightarrow{i_k^*} H^k(G) \rightarrow 0 \quad \text{for any } k > n, \\ H^n(Y) &\xrightarrow{\pi_n^*} H^n(X) \xrightarrow{i_n^*} H^n(G) \rightarrow 0, \\ 0 \rightarrow H_{2n-k}(G) &\rightarrow H^k(X) \xrightarrow{\beta_k^*} H^k(U) \rightarrow 0 \quad \text{for any } k < n. \end{aligned}$$

*Proof.* By [18, p. 84, 6\*] we know that  $H^k(Y, \text{Sing}(Y)) \cong H^k(X, G)$  for any  $k$ . Since  $\text{Sing}(Y)$  is finite, we also have  $H^k(Y, \text{Sing}(Y)) \cong H^k(Y)$  for  $k \geq 1$ . Therefore the long exact sequence of the pair:

$$\dots \rightarrow H^k(X, G) \rightarrow H^k(X) \xrightarrow{i_k^*} H^k(G) \rightarrow H^{k+1}(X, G) \rightarrow \dots$$

identifies, when  $k \geq 1$ , with the long exact sequence:

$$(13) \quad \dots \rightarrow H^k(Y) \xrightarrow{\pi_k^*} H^k(X) \xrightarrow{i_k^*} H^k(G) \rightarrow H^{k+1}(Y) \rightarrow \dots$$

In order to prove that the first two sequences are exact, it suffices to prove that  $i_k^*$  is surjective for any  $k \geq n$ . To this aim let  $L$  be a general hyperplane section of  $Y$ , and put  $Y_0 := Y \setminus L$ , and  $X_0 := \pi^{-1}(Y_0)$ . As before, we have a long exact sequence:

$$\dots \rightarrow H^k(Y_0) \rightarrow H^k(X_0) \rightarrow H^k(G) \rightarrow H^{k+1}(Y_0) \rightarrow \dots$$

and by the Deligne's Theorem [33, Proposition 4.23] we know that the pull-back maps  $H^k(X) \xrightarrow{i_k^*} H^k(G)$  and  $H^k(X_0) \rightarrow H^k(G)$  have the same image. Then we are done, because  $Y_0$  is affine, therefore  $H^{k+1}(Y_0) = 0$  for any  $k \geq n$  by stratified Morse Theory [21, p. 23-24].

In order to examine the last sequence, assume  $k < n$ . Then  $2n - k > n$ , and we just proved that the pull-back  $H^{2n-k}(X, G) \cong H^{2n-k}(Y) \rightarrow H^{2n-k}(X)$  is injective. By Poincaré Duality Theorem and Lefschetz Duality Theorem [31, p. 297] we have  $H^{2n-k}(X) \cong H_k(X)$  and  $H^{2n-k}(X, G) \cong H_k(U)$ . Therefore the push-forward  $H_k(U) \rightarrow H_k(X)$  is injective, hence the restriction  $H^k(X) \rightarrow H^k(U)$  is onto for any  $k < n$ . Now our assertion follows from (5).  $\square$

**Lemma 4.2.** *Fix an integer  $k$ , and let  $\gamma_k^* : H^k(Y, U) \rightarrow H^k(X, U)$  be the pull-back. Assume that  $\pi_k^* : H^k(Y) \rightarrow H^k(X)$  is injective. Then the following properties are equivalent.*

- (i)  $\gamma_k^*$  is injective;
- (ii)  $\Im(\alpha_{k-1}^*) = \Im(\beta_{k-1}^*)$ ;
- (iii)  $H^{k-1}(G) \rightarrow H^k(Y, U)$  is the zero map.

*Proof.* Consider the natural commutative diagram with exact rows:

$$\begin{array}{ccccccc} H^{k-1}(X) & \xrightarrow{\beta_{k-1}^*} & H^{k-1}(U) & \longrightarrow & H^k(X, U) & \longrightarrow & H^k(X) \\ \pi_{k-1}^* \uparrow & & \parallel & & \gamma_k^* \uparrow & & \pi_k^* \uparrow \\ H^{k-1}(Y) & \xrightarrow{\alpha_{k-1}^*} & H^{k-1}(U) & \longrightarrow & H^k(Y, U) & \longrightarrow & H^k(Y). \end{array}$$

If  $\gamma_k^*$  is injective then

$$\ker(H^{k-1}(U) \rightarrow H^k(X, U)) = \ker(H^{k-1}(U) \rightarrow H^k(Y, U)).$$

It follows that  $\Im(\alpha_{k-1}^*) = \Im(\beta_{k-1}^*)$  because  $\Im(\alpha_{k-1}^*) = \ker(H^{k-1}(U) \rightarrow H^k(Y, U))$  and  $\Im(\beta_{k-1}^*) = \ker(H^{k-1}(U) \rightarrow H^k(X, U))$ . Conversely, assume that  $\Im(\alpha_{k-1}^*) = \Im(\beta_{k-1}^*)$ , and fix any  $c \in \ker \gamma_k^*$ . Since  $\pi_k^*$  is injective, there exists some  $c' \in H^{k-1}(U)$  which maps to  $c$  via  $H^{k-1}(U) \rightarrow H^k(Y, U)$ . Since  $c \in \ker \gamma_k^*$ , a fortiori  $c'$  belongs to  $\Im(\beta_{k-1}^*)$ . Hence  $c' \in \Im(\alpha_{k-1}^*)$ , therefore  $c = 0$ . The equivalence of (i) with (iii) follows from (3).  $\square$

**Corollary 4.3.** *Let  $H_k(G) \rightarrow H^{2n-k}(G)$  be the map obtained composing the map  $H_k(G) \rightarrow H^{2n-k}(X)$  with the pull-back  $H^{2n-k}(X) \rightarrow H^{2n-k}(G)$ . Assume  $k \geq n$  and that  $\Im(\alpha_k^*) = \Im(\beta_k^*)$ . Then the map  $H_k(G) \rightarrow H^{2n-k}(G)$  is injective.*

*Proof.* By Lemma 4.1, Lemma 4.2, and (3), we deduce that the map  $H^k(X, U) \rightarrow H^k(G)$  is onto. Dualizing we get an injective map  $H_k(G) \rightarrow H_k(X, U)$ . We are done because by excision and Lefschetz Duality Theorem [31, p. 298] we have  $H_k(X, U) \cong H_k(D, \partial D) \cong H^{2n-k}(D) \cong H^{2n-k}(G)$ .  $\square$

**Corollary 4.4.** *We have:*

$$H^k(X) \cong \begin{cases} IH^k(Y) \oplus H^k(G) & \text{if } k > n \\ IH^k(Y) \oplus H_{2n-k}(G) & \text{if } k < n. \end{cases}$$

Moreover if  $\Im(\alpha_n^*) = \Im(\beta_n^*)$  then

$$H^n(X) \cong IH^n(Y) \oplus H^n(G).$$

*Proof.* In view of Lemma 4.1 we only have to examine the case  $k = n$ . Since  $\beta_n^* \circ \pi_n^* = \gamma_n^*$ , there exists a subspace  $P \subseteq \Im(\pi_n^*) \subseteq H^n(X)$  which is mapped isomorphically to  $\Im(\beta_n^*) = \Im(\alpha_n^*) = IH^n(Y)$  via  $\beta_n^*$ . In particular  $P \cap \ker \beta_n^* = \{0\}$ , and so  $H^n(X) = IH^n(Y) \oplus \ker \beta_n^*$ . On the other hand  $\ker \beta_n^* = \Im(H^n(X, U) \rightarrow H^n(X))$ . By Corollary 4.3 we know that the map  $H^n(X, U) \rightarrow H^n(X)$  is injective because so is the composite  $H^n(X, U) \cong H_n(G) \rightarrow H^n(X) \rightarrow H^n(G)$ . Therefore  $\ker \beta_n^* = \Im(H^n(X, U) \rightarrow H^n(X)) \cong H^n(X, U) \cong H_n(G) \cong H^n(G)$ .  $\square$

**Lemma 4.5.** *Assume that  $\Im(\alpha_k^*) = \Im(\beta_k^*)$  for any  $k \geq n$ . Then there is an injective map of complexes*

$$0 \rightarrow \mathcal{H}^\bullet \rightarrow \mathcal{J}^\bullet.$$

*Proof.* It is enough to prove that for any  $k$  there is a monomorphism of sheaves  $\mathcal{H}^k \hookrightarrow \ker(\mathcal{J}^k \rightarrow \mathcal{J}^{k+1})$ .

First we examine the case  $k \geq n$ .

To this aim, set  $\Gamma^\bullet := \Gamma(\mathcal{J}^\bullet)$  and denote by  $d^k : \Gamma^k \rightarrow \Gamma^{k+1}$  the differential. Then we have  $H^k(X) = H^k(\Gamma^\bullet)$ . By Lemma 4.1 any element  $a$  of  $H^k = H^k(G)$  can be lifted to an element  $c \in \ker d^k$ . We claim that any  $a \in H^k(G)$  can be lifted to an element  $b \in \ker d^k \subseteq \Gamma(\mathcal{J}^k)$  which is supported on  $\text{Sing}(Y)$ . Proving this claim amounts to show that any  $a \in H^k(G)$  can be lifted to an element  $b \in \ker d^k \subset \Gamma(\mathcal{J}^k) = \Gamma(\mathcal{I}^k)$  such that  $b|_U = 0 \in \Gamma(\mathcal{J}^k|_U)$ . But  $c|_U$  projects to a cohomology class living in  $\Im(H^k(X) \rightarrow H^k(U))$ . By our assumption we have

$$\Im(H^k(X) \xrightarrow{\beta_k^*} H^k(U)) \subseteq \Im(H^k(Y) \xrightarrow{\alpha_k^*} H^k(U)).$$

Since

$$H^k(Y) \cong H^k(Y, \text{Sing}(Y)) \cong H^k(X, G)$$

[18, p. 84, 6\*], we find

$$\Im(H^k(Y) \xrightarrow{\alpha_k^*} H^k(U)) = \Im(H^k(X, G) \rightarrow H^k(U)).$$

On the other hand we have

$$H^k(X, G) \cong H^k(X, \beta_! \mathbb{Q}_U)$$

[5, Theorem 12.1], [17, Remark 2.4.5, (ii)]. By definition of direct image with proper support [24, §2.6], [17, Definition 2.3.21], the sheaf  $\beta_! \mathbb{Q}_U$  identifies with the subsheaf of  $\mathbb{Q}_X$  consisting of sections with support contained in  $U$ . It follows there exists  $e_U \in \Gamma(\mathcal{J}^{k-1}|_U)$  and  $g \in \Gamma(\mathcal{J}^k)$  supported in  $U$  such that

$$c|_U - d^{k-1}(e_U) = g|_U.$$

Moreover there exists  $e \in \Gamma(\mathcal{J}^{k-1})$  with  $e|_U = e_U$ , because  $\mathcal{J}^{k-1}$  is injective (hence flabby). We conclude that the section

$$c - g - d^{k-1}(e) \in \Gamma(\mathcal{J}^k)$$

is supported on  $\text{Sing}(Y)$ . Our claim is proved because  $g + d^{k-1}(e) \in \Gamma(\mathcal{J}^k)$  vanishes in  $H^k(G)$ . To conclude the proof in the case  $k \geq n$ , fix a basis  $a_r \in H^k = H^k(G)$  and lift any  $a_r$  to a  $b_r \in \ker d^k \subseteq \Gamma(\mathcal{J}^k)$  as in the claim. We get an isomorphism between  $H^k(G)$  and a subspace of  $\Gamma(\mathcal{J}^k)$  consisting of sections supported on  $\text{Sing}(Y)$ . We

are done because such an isomorphism projects to a monomorphism of sheaves  $\mathcal{H}^k \hookrightarrow \ker(J^k \rightarrow J^{k+1})$ .

Now assume  $k < n$ .

By Lemma 4.1 any element  $a$  of  $H^k = H_{2n-k}(G) \subseteq H^k(X)$  can be lifted to an element  $c \in \ker d^k$ . Since  $a$  restricts to 0 in  $H^k(U)$ , then there exists  $e \in \Gamma(\mathcal{J}^{k-1}|_U)$  such that  $c|_U = d_U^{k-1}(e)$ . Since  $\mathcal{J}^{k-1}$  is flabby we may assume  $e \in \Gamma(\mathcal{J}^{k-1})$ . Therefore  $b := c - d^{k-1}(e) \in \Gamma(\mathcal{J}^k)$  represents  $a$  and is supported on  $\text{Sing}(Y)$ . As in the case  $k \geq n$ , applying this argument to a basis of  $H^k = H_{2n-k}(G)$ , we define a monomorphism of sheaves  $\mathcal{H}^k \hookrightarrow \ker(\mathcal{J}^k \rightarrow \mathcal{J}^{k+1})$ .  $\square$

With the same assumption as in Lemma 4.5, let  $\mathcal{K}^\bullet$  be the cokernel of the inclusion  $0 \rightarrow \mathcal{H}^\bullet \rightarrow \mathcal{J}^\bullet$ :

$$0 \rightarrow \mathcal{H}^\bullet \rightarrow \mathcal{J}^\bullet \rightarrow \mathcal{K}^\bullet \rightarrow 0.$$

All the sheaves of these complexes are injective. Previous sequence gives rise to a long exact sequence of sheaf cohomology:

$$\dots \rightarrow \mathcal{H}^k \rightarrow \mathcal{H}^k(\mathcal{J}^\bullet) \rightarrow \mathcal{H}^k(\mathcal{K}^\bullet) \rightarrow \dots,$$

and for any  $k \geq 1$  these sheaves are supported on  $\text{Sing}(Y)$ .

**Proposition 4.6.** *For any  $k$  the sequence*

$$0 \rightarrow \mathcal{H}^k \rightarrow \mathcal{H}^k(\mathcal{J}^\bullet) \rightarrow \mathcal{H}^k(\mathcal{K}^\bullet) \rightarrow 0$$

*is exact.*

*Proof.* It suffices to prove that the map  $H_y^k \rightarrow \mathcal{H}^k(\mathcal{J}^\bullet)_y$  is injective for any  $y \in \text{Sing}(Y)$  and any  $k > 0$ . If  $k \geq n$  this is obvious because  $H^k(\mathcal{J}^\bullet)_y = H^k(G_y) = H_y^k$ . When  $1 \leq k < n$  we have  $H_y^k = H_{2n-k}^k(G_y)$ . And the map  $H_{2n-k}(G_y) \rightarrow H^k(\mathcal{J}^\bullet)_y = H^k(G_y)$  is injective by Corollary 4.3.  $\square$

**Lemma 4.7.** *Let  $0 \rightarrow \mathcal{H}^\bullet \xrightarrow{f} \mathcal{J}^\bullet \xrightarrow{g} \mathcal{K}^\bullet \rightarrow 0$  be an exact sequence of complexes of sheaves. Assume that  $\mathcal{H}^\bullet$  is a complex of injective sheaves with vanishing differential  $d_{\mathcal{H}^\bullet}^k = 0$  for any  $k$ . The following properties are equivalent.*

(i) *The sequence coming from the cohomology long exact sequence:*

$$(14) \quad 0 \rightarrow \mathcal{H}^k(\mathcal{H}^\bullet) \rightarrow \mathcal{H}^k(\mathcal{J}^\bullet) \rightarrow \mathcal{H}^k(\mathcal{K}^\bullet) \rightarrow 0$$

*is exact for any  $k$ .*

(ii) *There is a complex map  $s^\bullet : \mathcal{K}^\bullet \rightarrow \mathcal{J}^\bullet$  such that  $g^\bullet \circ s^\bullet = \text{id}_{\mathcal{K}^\bullet}$ .*

*Proof.* We only have to prove that (i) implies (ii).

Since  $\mathcal{H}^0$  is injective, the exact sequence  $0 \rightarrow \mathcal{H}^0 \rightarrow \mathcal{J}^0 \rightarrow \mathcal{K}^0 \rightarrow 0$  admits a section  $s^0 : \mathcal{K}^0 \rightarrow \mathcal{J}^0$ , with  $g^0 \circ s^0 = \text{id}_{\mathcal{K}^0}$ . Therefore we may construct  $s^\bullet = \{s^i\}_{i \geq 0}$  using induction on  $i$ . Assume  $i \geq 0$  and that there are sections  $s^0, \dots, s^i$ , with  $s^h : \mathcal{K}^h \rightarrow \mathcal{J}^h$ ,  $g^h \circ s^h = \text{id}_{\mathcal{K}^h}$ , and  $s^h \circ d_{\mathcal{K}^\bullet}^{h-1} = d_{\mathcal{J}^\bullet}^{h-1} \circ s^{h-1}$  for any  $0 \leq h \leq i$ . As before, since  $\mathcal{H}^{i+1}$  is injective and the sequence  $0 \rightarrow \mathcal{H}^{i+1} \rightarrow \mathcal{J}^{i+1} \rightarrow \mathcal{K}^{i+1} \rightarrow 0$  is exact, there exists a section  $\sigma^{i+1} : \mathcal{K}^{i+1} \rightarrow \mathcal{J}^{i+1}$ , with

$g^{i+1} \circ \sigma^{i+1} = \text{id}_{\mathcal{K}^{i+1}}$ . A priori it may happen that  $\sigma^{i+1} \circ d_{\mathcal{K}^\bullet}^i$  is different from  $d_{\mathcal{J}^\bullet}^i \circ s^i$ , so we have to modify  $\sigma^{i+1}$ . To this purpose set:

$$\delta := \sigma^{i+1} \circ d_{\mathcal{K}^\bullet}^i - d_{\mathcal{J}^\bullet}^i \circ s^i \in \text{Hom}(\mathcal{K}^i, \mathcal{J}^{i+1}).$$

Since

$$g^{i+1} \circ \delta = g^{i+1} \circ \sigma^{i+1} \circ d_{\mathcal{K}^\bullet}^i - g^{i+1} \circ d_{\mathcal{J}^\bullet}^i \circ s^i = d_{\mathcal{K}^\bullet}^i - d_{\mathcal{K}^\bullet}^i = 0,$$

it follows that

$$(15) \quad \Im(\delta) \subseteq \mathcal{H}^{i+1}.$$

Moreover, since (14) is exact, the map  $g^i$  sends  $\ker d_{\mathcal{J}^\bullet}^i$  onto  $\ker d_{\mathcal{K}^\bullet}^i$ . This in turn implies that the section  $s^i$  sends  $\ker d_{\mathcal{K}^\bullet}^i$  in  $\ker d_{\mathcal{J}^\bullet}^i$ , because

$$\ker g^i = \Im(f^i) \cong \mathcal{H}^i = \mathcal{H}^i(\mathcal{H}^\bullet) \subseteq \ker d_{\mathcal{J}^\bullet}^i.$$

in view of the assumption  $d_{\mathcal{H}^\bullet}^i = 0$ . We deduce that:

$$(16) \quad \ker d_{\mathcal{K}^\bullet}^i \subseteq \ker \delta,$$

and from (15) and (16) we get

$$\delta \in \text{Hom}(\mathcal{K}^i / \ker d_{\mathcal{K}^\bullet}^i, \mathcal{H}^{i+1}).$$

Since  $\mathcal{H}^{i+1}$  is injective, we may extend  $\delta$  to a map  $\tilde{\delta} \in \text{Hom}(\mathcal{K}^{i+1}, \mathcal{H}^{i+1})$  such that

$$(17) \quad \tilde{\delta} \circ d_{\mathcal{K}^\bullet}^i = \delta.$$

We have

$$\tilde{\delta} \in \text{Hom}(\mathcal{K}^{i+1}, \mathcal{J}^{i+1})$$

because  $\mathcal{H}^{i+1}$  maps to  $\mathcal{J}^{i+1}$  via  $f^{i+1}$ . Now we define:

$$s^{i+1} := \sigma^{i+1} - \tilde{\delta}.$$

From (17) it follows that

$$s^{i+1} \circ d_{\mathcal{K}^\bullet}^i = d_{\mathcal{J}^\bullet}^i \circ s^i,$$

and since  $\Im(\tilde{\delta}) \subseteq \mathcal{H}^{i+1}$  we also have

$$g^{i+1} \circ s^{i+1} = \text{id}_{\mathcal{K}^{i+1}}.$$

□

## 5. PROOF OF THEOREM 3.1

As we noticed in Section 3, by [32, Theorem 1.11, p. 518] one knows that the Decomposition Theorem implies (ii). Therefore we only have to prove that (ii) implies (i).

In view of Lemma 4.1 and Lemma 4.2 we have  $\Im(\alpha_k^*) = \Im(\beta_k^*)$  for any  $k \geq n$ . From Lemma 4.5, Proposition 4.6, and Lemma 4.7, we get:

$$(18) \quad R\pi_* \mathbb{Q}_X = \mathcal{J}^\bullet = \mathcal{K}^\bullet \oplus \mathcal{H}^\bullet.$$

Hence we only have to prove that

$$\mathcal{K}^\bullet \cong IC_Y[-n],$$

where  $IC_Y^\bullet = IC_Y^{top}[-n]$  denotes the intersection cohomology complex of  $Y$  [17, p. 156]. Observe that the restriction  $\alpha^{-1}\mathcal{K}^\bullet$  of  $\mathcal{K}^\bullet$  to  $U$  is  $\mathbb{Q}_U$ , and that, by (18), we have  $\mathcal{K}^\bullet \in D_c^b(Y)$  [17, p. 81-82]. Therefore  $\mathcal{K}^\bullet[n]$  is an extension of  $\mathbb{Q}_U[n]$  [17, p. 134]. So to prove that  $\mathcal{K}^\bullet \cong IC_Y[-n]$  it suffices to prove that  $\mathcal{K}^\bullet[n] \cong \alpha_{!*}\mathbb{Q}_U[n]$ , i.e. that  $\mathcal{K}^\bullet[n]$  is the intermediary extension of  $\mathbb{Q}_U[n]$  [17, p.156 and p.135]. By [17, Proposition 5.2.8, p. 135], this in turn reduces to prove that for any  $y \in \text{Sing}(Y)$  the following two conditions hold true ( $i_y : \{y\} \rightarrow Y$  denotes the inclusion):

- (a)  $\mathcal{H}^{k i_y^{-1}} \mathcal{K}^\bullet[n] = 0$  for any integer  $k \geq 0$ ;
- (b)  $\mathcal{H}^{k i_y^!} \mathcal{K}^\bullet[n] = 0$  for any integer  $k \leq 0$ .

As for condition (a) we notice that [17, p.130]:

$$\mathcal{H}^k i_y^{-1} \mathcal{K}^\bullet[n] = \mathcal{H}^k(\mathcal{K}^\bullet[n])_y = \mathcal{H}^{k+n}(\mathcal{K}^\bullet)_y,$$

and  $\mathcal{H}^{k+n}(\mathcal{K}^\bullet)_y = 0$  because  $\mathcal{J}^\bullet = \mathcal{K}^\bullet \oplus \mathcal{H}^\bullet$ , and  $\mathcal{H}^{k+n}(\mathcal{J}^\bullet)_y = H^{k+n}(G_y) = \mathcal{H}^{k+n}(\mathcal{H}^\bullet)_y$  for  $k \geq 0$ .

For the condition (b), first notice that combining (18) with Remarks 2.3 and 2.7, we deduce that  $\mathcal{K}^\bullet[n]$  is self-dual. Therefore condition (b) reduces to (a). In fact we have [17, p. 130, proof of Lemma 5.1.15]:

$$\mathcal{H}^k i_y^! \mathcal{K}^\bullet[n] = \mathcal{H}^{-k}(i_y^{-1} D(\mathcal{K}^\bullet[n]))^\vee = \mathcal{H}^{-k}(i_y^{-1}(\mathcal{K}^\bullet[n]))^\vee = \mathcal{H}^{-k+n}(\mathcal{K}^\bullet)_y^\vee = 0$$

because  $k \leq 0$ .

*Remark 5.1.* (i) If  $n = 2$  then the map  $H^{k-1}(G) \rightarrow H^k(Y, U)$  vanishes for any  $k \geq n + 2$  for trivial reasons. In view of the connectivity of the link, combining Remark 2.2 with Lemma 4.1 and Lemma 4.2, we see that this holds true also when  $Y$  is locally complete intersection. Therefore, either when  $n = 2$  or when  $Y$  is locally complete intersection, in order to deduce the decomposition (i) in Theorem 3.1, one only has to check that the map  $H^n(G) \rightarrow H^{n+1}(Y, U)$  is the zero map. On the other hand, the vanishing of the map  $H^n(G) \rightarrow H^{n+1}(Y, U)$  is equivalent to require that the natural map  $H_n(G) \rightarrow H^n(G) \cong H_n(G)^\vee$  is onto (compare with (3), (5), and Corollary 4.3). Since  $H_n(G)$  is contained in  $H_n(X)$  via push-forward (Lemma 4.1), it follows that the map  $H_n(G) \rightarrow H^n(G) \cong H_n(G)^\vee$  is onto if and only if  $H_n(G)$  is a non degenerate subspace of  $H_n(X)$  with respect to the natural intersection form  $H_n(X) \times H_n(X) \rightarrow H_0(X) \cong \mathbb{Q}$ . By Mumford's Theorem [23], [29] we know this holds true when  $Y$  is a normal surface. Therefore, *in the case  $Y$  is a normal surface (or when  $2 \dim G < n$ ), our Theorem 3.1 gives a new and simplified proof of the Decomposition Theorem for  $\pi : X \rightarrow Y$ .*

(ii) Assume that  $\pi : X \rightarrow Y$  is the blowing-up of  $Y$  along  $\text{Sing}(Y)$ , with smooth and connected fibres. By Poincaré Duality we have  $H_{2n-k}(G_y) \cong H^{k-2}(G_y)$  for any  $y \in \text{Sing}(Y)$ . It follows that  $H^k(X, U) \cong H_{2n-k}(G) \cong \bigoplus_{y \in \text{Sing}(Y)} H_{2n-k}(G_y) \cong \bigoplus_{y \in \text{Sing}(Y)} H^{k-2}(G_y)$ . Hence the map  $H^k(X, U) \rightarrow H^k(G)$  identifies with the map  $\bigoplus_{y \in \text{Sing}(Y)} H^{k-2}(G_y) \rightarrow \bigoplus_{y \in \text{Sing}(Y)} H^k(G_y)$  given, on each summand  $H^{k-2}(G_y) \rightarrow H^k(G_y)$ , by the self-intersection formula, i.e. by the cup-product with the first Chern class  $c_1(N_y) \in H^2(G_y)$  of the normal bundle  $N_y$  of  $G_y$  in  $X$ . Since  $\pi$  is the

blowing-up along the finite set  $\text{Sing}(Y)$ , the dual normal bundle  $N_y^\vee \cong \mathcal{O}_{G_y}(1)$  is ample for any  $y \in \text{Sing}(Y)$ . By Hard Lefschetz Theorem it follows that the map  $H^{k-2}(G_y) \rightarrow H^k(G_y)$  is onto for any  $k \geq n$ , and so also the map  $H^k(X, U) \rightarrow H^k(G)$  is. By (3), this implies the vanishing of the map  $H^k(G) \rightarrow H^{k+1}(Y, U)$ . Therefore, also in this case our Theorem 3.1 gives a new and simplified proof of the Decomposition Theorem for  $\pi$ .

(iii) More generally, assume only that the fibres of  $\pi : X \rightarrow Y$  are smooth and connected, so that  $\pi$  is not necessarily the blowing-up along  $\text{Sing}(Y)$ . Using the extension of the Hard Lefschetz Theorem to bundles of higher rank due to Bloch and Gieseker [3], [25], with a similar argument as before one proves that *if the dual normal bundle  $N_y^\vee$  of  $G_y$  in  $X$  is ample for any  $y \in \text{Sing}(Y)$ , then the map  $H^k(G) \rightarrow H^{k+1}(Y, U)$  vanishes for any  $k \geq n$* . In fact, set  $h_y := \dim X - \dim G_y$  for any  $y \in \text{Sing}(Y)$ . Now the map  $H^k(X, U) \rightarrow H^k(G)$  identifies with the map  $\bigoplus_{y \in \text{Sing}(Y)} H^{k-2h_y}(G_y) \rightarrow \bigoplus_{y \in \text{Sing}(Y)} H^k(G_y)$  given, on each summand  $H^{k-2h_y}(G_y) \rightarrow H^k(G_y)$ , by the cup-product with the top Chern class  $c_{h_y}(N_y) = (-1)^{h_y} c_{h_y}(N_y^\vee) \in H^{2h_y}(G_y)$  of the normal bundle  $N_y$  of  $G_y$  in  $X$ . And such a map is onto for  $k \geq n$  by the quoted extension of the Hard Lefschetz Theorem, because  $N_y^\vee$  is ample. We refer to [15, Proposition 2.12 and proof] for examples of resolution of singularities verifying previous assumptions.

## 6. PROOF OF THEOREM 3.2

(i)  $\implies$  (ii) By Lemma 4.1 and Lemma 4.2 we have  $\mathfrak{S}(\alpha_k^*) = \mathfrak{S}(\beta_k^*)$  for any  $k$ . Let  $y_1, \dots, y_a, y_{a+1}, \dots, y_b$  be a basis of  $H^k(Y)$  such that  $\alpha_k^* y_1, \dots, \alpha_k^* y_a$  is a basis for  $\mathfrak{S}(\alpha_k^*) = \mathfrak{S}(\beta_k^*)$ , and  $y_{a+1}, \dots, y_b$  a basis for  $\ker \alpha_k^*$ . Since  $\pi_k^*(\ker \alpha_k^*) \subseteq \ker \beta_k^*$ , we may extend  $\pi_k^* y_{a+1}, \dots, \pi_k^* y_b$  to a basis  $\pi_k^* y_{a+1}, \dots, \pi_k^* y_b, x_{b+1}, \dots, x_c$  of  $\ker \beta_k^*$ . Then  $\pi_k^* y_1, \dots, \pi_k^* y_a, \pi_k^* y_{a+1}, \dots, \pi_k^* y_b, x_{b+1}, \dots, x_c$  is a basis for  $H^k(X)$ . Define  $\theta_k : H^k(X) \rightarrow H^k(Y)$  setting  $\theta_k(\pi_k^*(y_i)) := y_i$ , and  $\theta_k(x_i) := 0$ . Then  $\theta_\bullet$  is a natural morphism.

(ii)  $\implies$  (i) The existence of a natural morphism implies that  $\pi_k^*$  is injective and  $\mathfrak{S}(\beta_k^*) \subseteq \mathfrak{S}(\alpha_k^*)$  for any  $k$ . Since in general we have  $\mathfrak{S}(\alpha_k^*) \subseteq \mathfrak{S}(\beta_k^*)$  it follows that  $\mathfrak{S}(\alpha_k^*) = \mathfrak{S}(\beta_k^*)$  for any  $k$ . By Lemma 4.1 and Lemma 4.2 we get (i).

(ii)  $\implies$  (iv) Since  $\pi_k^*$  is injective for any  $k$ , using (13) we get a short exact sequence:

$$0 \rightarrow H^k(Y) \xrightarrow{\pi_k^*} H^k(X) \xrightarrow{i_k^*} H^k(G) \rightarrow 0$$

for any  $k \geq 1$ . In particular, for any  $k \geq 1$ , we have

$$(19) \quad H^k(X) \cong H^k(Y) \oplus H^k(G).$$

On the other hand, since  $\theta_k \circ \pi_k^* = \text{id}_{H^k(Y)}$ , the short exact sequence

$$0 \rightarrow \ker \theta_k \rightarrow H^k(X) \xrightarrow{\theta_k} H^k(Y) \rightarrow 0$$

admits  $\pi_k^*$  as a section. It follows another decomposition:

$$(20) \quad H^k(X) = \pi_k^* H^k(Y) \oplus \ker \theta_k.$$

Comparing (19) with (20) we see that

$$\ker \theta_k \cong H^k(G)$$

for any  $k \geq 1$ . On the other hand since  $\alpha_k^* \circ \theta_k = \beta_k^*$  we have

$$(21) \quad \ker \theta_k \subseteq \ker(H^k(X) \xrightarrow{\beta_k^*} H^k(U)) = \Im(H^k(X, U) \rightarrow H^k(X)).$$

Since  $H^k(X, U) \cong H_{2n-k}(G)$  it follows that

$$(22) \quad \dim H^k(G) \leq \dim H_{2n-k}(G)$$

for any  $k \geq 1$ . By Universal-coefficient formula [31, p. 248] we deduce that, for  $1 \leq k \leq 2n-1$ ,

$$(23) \quad \ker \theta_k \cong H^k(G) \cong H_{2n-k}(G).$$

Taking into account that  $\Im(\alpha_n^*) = \Im(\beta_n^*)$ , combining (19), (23) and Corollary 4.4, it follows that  $\dim H^k(Y) = \dim IH^k(Y)$  for any  $k$ . Therefore, by (11), it suffices to prove that  $\alpha_k^* : H^k(Y) \rightarrow H^k(U)$  is surjective for any  $k < n$ . To this purpose notice that, for  $k < n$ ,  $\beta_k^*$  is surjective by Lemma 4.1. This implies that also  $\alpha_k^*$  is by (20) and (21) (compare with diagram (2)).

(iv)  $\implies$  (vii) Since intersection cohomology verifies Poincaré Duality [17, p. 158], we have:

$$H^h(Y) = IH^h(Y) = (IH^{2(m+1)-h}(Y))^\vee = (H^{2(m+1)-h}(Y))^\vee = H_{2(m+1)-h}(Y).$$

(vii)  $\implies$  (iv) This follows from (11) and Remark 2.1.

(v)  $\iff$  (vi)  $\iff$  (vii) By [28, Theorem 2, Lemma 2, Lemma 3] we know that the duality morphism is an isomorphism if and only if  $Y$  is a  $\mathbb{Q}$ -cohomology manifold, which is equivalent to say that  $Y$  is a  $\mathbb{Q}$ -intersection cohomology manifold by [27, Theorem 1.1] (compare also with [4]).

(vii)  $\implies$  (ii) Denote by  $d_k^Y : H^k(Y) \rightarrow H_{2n-k}(Y)$  the duality isomorphism, by  $d_k^X : H^k(X) \cong H_{2n-k}(X)$  the Poincaré Duality isomorphism, by  $\pi_{*,k} : H_{2n-k}(X) \rightarrow H_{2n-k}(Y)$  the push-forward. Set  $\theta_k : H^k(X) \rightarrow H^k(Y)$  with

$$\theta_k := (d_k^Y)^{-1} \circ \pi_{*,k} \circ d_k^X.$$

Then  $\theta_\bullet$  is a natural morphism.

(iii)  $\iff$  (ii) We only have to prove that (ii) implies (iii). This follows from Remark 2.6 because  $Y$  is a  $\mathbb{Q}$ -cohomology manifold.

(ii)  $\implies$  (viii) Since  $Y$  is a  $\mathbb{Q}$ -intersection cohomology manifold, combining (23) with Theorem 3.1 we get:

$$R\pi_* \mathbb{Q}_X \cong \mathbb{Q}_Y \oplus \mathcal{H}^\bullet \cong \mathbb{Q}_Y \oplus \bigoplus_{k \geq 1} R^k \pi_* \mathbb{Q}_X[-k].$$

(viii)  $\implies$  (ii) See Remark 3.3, (i).



(ii)  $\iff$  (ix) By [27, Theorem 1.1] we deduce that  $Y$  is a  $\mathbb{Q}$ -intersection cohomology manifold if and only if for any  $y \in \text{Sing}(Y)$  the link  $\partial B_y$  has the same  $\mathbb{Q}$ -homology type as a sphere  $S^{2n-1}$ . On the other hand, via deformation to the normal cone, we may identify  $\partial B_y$  with the link of the vertex of the projective cone over  $G_y \subseteq \mathbb{P}^{N-1}$ . Restricting the Hopf bundle  $S^{2N-1} \rightarrow \mathbb{P}^{N-1}$  to  $G_y$ , we obtain an  $S^1$ -bundle  $\partial B_y \rightarrow G_y$  inducing the Thom-Gysin sequence [31, p. 260]

$$\cdots \rightarrow H^k(G_y) \rightarrow H^k(\partial B_y) \rightarrow H^{k-1}(G_y) \rightarrow H^{k+1}(G_y) \rightarrow H^{k+1}(\partial B_y) \rightarrow \cdots$$

And this sequence implies that  $\partial B_y$  has the same  $\mathbb{Q}$ -homology type as a sphere  $S^{2n-1}$  if and only if  $H^\bullet(G_y) \cong H^\bullet(\mathbb{P}^{n-1})$ .

*Remark 6.1.* By (22) it follows that  $h_2(G) \leq h_{2n-2}(G)$ . Therefore if  $Y$  is a  $\mathbb{Q}$ -cohomology manifold then  $\dim G = 0$  or  $\dim G = n - 1$ .

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